The Nonlinear Pendulum

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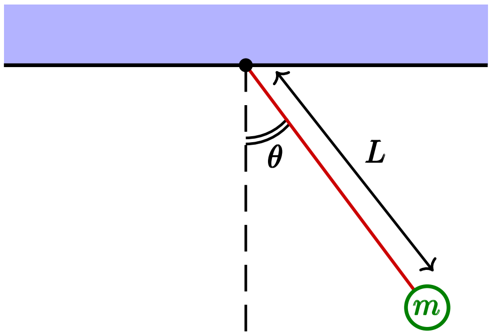
Consider the motion of a pendulum: a material point of mass m suspended on an inextensible string of length L:



Newton's second law[1] yields an equation of motion of the pendulum in terms of the angular displacement $\theta(t)$:

$$\begin{cases} \ddot{\theta}(t) + \omega^2 \sin(\,\theta(t)\,) &= 0 \\ \theta(0) &= \theta_0 \in \mathbb{R} \\ \dot{\theta}(0) &= 0 \end{cases}$$

where, for simplicity, we have considered the initial angular velocity $\dot{\theta}(t)$ of the pendulum to be zero (i.e. the pendulum is dropped from rest) and we have defined

$$\omega \equiv \sqrt{\frac{g}{L}}.$$

Note that values of $\theta_0, \theta(t)$ must lie in the interval $[-\pi/2, \pi/2]$.

To solve the ODE, we multiply by $\dot{\theta}(t)$:

$$\dot{\theta}(t)\,\ddot{\theta}(t) + \omega^2\dot{\theta}(t)\,\sin(\,\theta(t)\,) = \frac{d}{dt}\left[\frac{1}{2}\dot{\theta}(t)^2 - \omega^2\cos(\,\theta(t)\,)\right] = 0$$

using the chain rule. Integrating this equation over $[0, t]$ yields:

$$\left[\frac{1}{2}\dot{\theta}(t)^2 - \omega^2\cos(\,\theta(t)\,)\right]_0^t = 0 \implies \frac{1}{2}\dot{\theta}(t)^2 - \omega^2\cos(\,\theta(t)\,) + \omega^2\cos(\theta_0) = 0$$

$$\implies \dot{\theta}(t)^2 = 2\omega^2\Big[\,\cos(\,\theta(t)\,) - \cos(\theta_0)\Big].$$

[1]Consider tangential forces: $F = -mg\sin(\,\theta(t)\,)$ due to the weight of the mass.

Using the double angle formula $\cos(\,\theta(t)\,) = 1 - 2\sin^2(\theta(t)/2)$, we write:

$$\begin{aligned}
\cos(\,\theta(t)\,) - \cos(\theta_0) \;&=\; 1 - 2\sin^2(\theta(t)/2) - \cos(\theta_0) \;=\; \Big(1 - \cos(\theta_0\,)\Big)\left(1 - \frac{2\sin^2(\theta(t)/2)}{1 - \cos(\theta_0)}\right) \\
&=\; 2\sin^2(\theta_0/2)\left(1 - \frac{2\sin^2(\theta(t)/2)}{1 - \cos(\theta_0)}\right) \;=\; 2\sin^2(\theta_0/2)\left(1 - \frac{\sin^2(\theta(t)/2)}{\sin^2(\theta_0/2)}\right) \\
&=\; 2k\left(1 - \frac{\sin^2(\theta(t)/2)}{k^2}\right)
\end{aligned}$$

where we have introduced the variable

$$k \;\equiv\; \sin(\theta_0/2)$$

and used $1 - \cos(\theta_0) = 2\sin^2(\theta_0/2)$. Note the value of k depends upon the initial condition and must lie in the interval $[0, \pi/4]$ (since $\theta \in [0, \pi/2]$). Thus, we have the differential equation:

$$\dot{\theta}(t)^2 \;=\; 4\omega^2 k^2\left(1 - \frac{\sin^2(\theta(t)/2)}{k^2}\right) \quad\Longrightarrow\quad \frac{\dot{\theta}(t)}{2k} \;=\; \omega\sqrt{1 - \frac{\sin^2(\theta(t)/2)}{k^2}}.$$

We wish to find the period of oscillation of the pendulum from this differential equation. We note from the initial conditions the pendulum is at an angle $\theta = \theta_0$ and at some (later) time, we must have $\theta = 0$ when it passes the lowest point of its arc (we must now assume $\theta_0 > 0$). The time it takes the pendulum to travel between these two points is equivalent to one quarter of the time period, T. That is:

$$\frac{T}{4} \;=\; \int_{\theta=0}^{\theta_0} \frac{dt}{d\theta}\,d\theta \quad\Longrightarrow\quad T \;=\; 4\int_{\theta=0}^{\theta_0} \frac{d\theta}{\dot{\theta}(t)} \;=\; 4\int_{\alpha=0}^{\pi/2} \frac{d\alpha}{\dot{\alpha}(t)}.$$

in terms of a new angle variable $\alpha(t)$ defined by:

$$\sin(\,\alpha(t)\,) \;\equiv\; \frac{\sin(\theta(t)/2)}{k} \;=\; \frac{\sin(\theta(t)/2)}{\sin(\theta_0/2)}$$

which, as $\theta(t)$ varies over the interval $[0, \theta_0]$, varies over $[0, \pi/2]$. Differentiating the relation with respect to t gives:

$$\dot{\alpha}(t)\,\cos(\,\alpha(t)\,) \;=\; \frac{\dot{\theta}(t)\,\cos(\theta(t)/2)}{2k}$$

or using $\cos(\theta(t)/2) = \sqrt{1 - \sin^2(\theta(t)/2)} = \sqrt{1 - k^2\sin^2(\,\alpha(t)\,)}$:

$$\frac{\dot{\theta}(t)}{2k} \;=\; \frac{\dot{\alpha}(t)\,\cos(\,\alpha(t)\,)}{\cos(\theta(t)/2)} \;=\; \frac{\dot{\alpha}(t)\,\cos(\,\alpha(t)\,)}{\sqrt{1 - k^2\sin^2(\,\alpha(t)\,)}}.$$

Thus, the differential equation becomes:

$$\frac{\dot{\alpha}(t)\,\cos(\,\alpha(t)\,)}{\sqrt{1 - k^2\sin^2(\,\alpha(t)\,)}} \;=\; \omega\sqrt{1 - \sin^2(\,\alpha(t)\,)} \;=\; \omega\,\cos(\,\alpha(t)\,)$$

or after rearranging:

$$\dot{\alpha}(t) = \omega \sqrt{1 - k^2 \sin^2(\,\alpha(t)\,)}.$$

Hence the time period can be written:

$$T = \frac{4}{\omega} \int_{\alpha=0}^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2(\alpha)}} = \frac{4\,K(k)}{\omega}$$

in terms of the complete elliptic integral of the first kind K with modulus k. Recall that $k \in (0, \pi/4)$ which is a valid parameter range for the elliptic integral and implies the integrand will not go singular. Since the (generalised binomial) series[2]

$$\begin{aligned}\frac{1}{\sqrt{1-x}} &= (1-x)^{-\frac{1}{2}} = \sum_{j=0}^{\infty} \binom{-1/2}{j} (-x)^j = 1 + \sum_{j=1}^{\infty} \binom{-1/2}{j} (-x)^j \\ &= 1 + \sum_{j=1}^{\infty} (-1)^j \left(\prod_{i=1}^{j} \frac{1/2 - i}{i} \right) x^j = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\prod_{i=1}^{j} \frac{1 - 2i}{2i} \right) x^j\end{aligned}$$

is convergent[3] for $x \in (0, 1)$, we may write

$$\frac{1}{\sqrt{1 - k^2 \sin^2(\alpha)}} = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\prod_{i=1}^{j} \frac{1 - 2i}{2i} \right) k^{2j} \sin^{2j}(\alpha) \tag{1}$$

since $k^2 \sin(\alpha) \in (0, \pi^2/16) \subset (0, 1)$ for all α. Furthermore

$$\begin{aligned}\int_{\alpha=0}^{\pi/2} \sin^{2n}(\alpha)\, d\alpha &= \left(\frac{2n-1}{2n}\right) I_{2n-2} = \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right) I_{2n-4} = \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right) \cdots \left(\frac{1}{2}\right) I_0 \\ &= \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}\end{aligned}$$

using the reduction formula[4]

$$I_n \equiv \int_{\alpha=0}^{\pi/2} \sin^n(\alpha)\, d\alpha = \left(\frac{n-1}{n}\right) I_{n-2}, \qquad I_0 = \int_{\alpha=0}^{\pi/2} d\alpha = \frac{\pi}{2}.$$

Thus, due to the uniform convergence[5] of the series (1), we have a power series expansion of the

[2] Here we use the generalised binomial coefficients $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k(k-1)\cdots 1} = \prod_{i=1}^{k} \frac{\alpha - i + 1}{i}$.

[3] This series converges absolutely on $(0, 1)$ and uniformly on $(0, \rho]$ for $0 < \rho < 1$ by the Weierstrass M-test.

[4] The reduction formula can be proved by a simple application of integration by parts

[5] This permits us to interchange summation and integration.

complete elliptic integral of the first kind:

$$
\begin{aligned}
K(k) &= \int_{\alpha=0}^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2(\alpha)}} = \int_{\alpha=0}^{\pi/2} \left(1 + \sum_{j=1}^{\infty} (-1)^j \left(\prod_{i=1}^{j} \frac{1-2i}{2i} \right) k^{2j} \sin^{2j}(\alpha) \right) d\alpha \\
&= \int_{\alpha=0}^{\pi/2} d\alpha + \sum_{j=1}^{\infty} (-1)^j \left(\prod_{i=1}^{j} \frac{1-2i}{2i} \right) k^{2j} \int_{\alpha=0}^{\pi/2} \sin^{2j}(\alpha) d\alpha \\
&= \frac{\pi}{2} + \sum_{j=1}^{\infty} (-1)^j \left(\prod_{i=1}^{j} \frac{1-2i}{2i} \right) k^{2j} \cdot \left(\frac{\pi}{2} \prod_{i=1}^{j} \frac{2i-1}{2i} \right) \\
&= \frac{\pi}{2} + \frac{\pi}{2} \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} (-1) \right) \left(\prod_{i=1}^{j} \frac{1-2i}{2i} \right) \left(\prod_{i=1}^{j} \frac{2i-1}{2i} \right) k^{2j} = \frac{\pi}{2} \left[1 + \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} \frac{2i-1}{2i} \right)^2 k^{2j} \right].
\end{aligned}
$$

It can be checked (using the ratio test, for example) that the series in the expansion above converges absolutely for all $k \in [0, 1)$. Thus, we may write the time period of our pendulum as

$$
T = \frac{4K(k)}{\omega} K = \frac{2\pi}{\omega} \left[1 + \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} \frac{2i-1}{2i} \right)^2 k^{2j} \right]
$$

or, after restoring variables and expanding:

$$
T = 2\pi \sqrt{\frac{L}{g}} \left[1 + \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} \frac{2i-1}{2i} \right)^2 \sin^{2j} \left(\frac{\theta_0}{2} \right) \right]
$$

using $\omega = \sqrt{g/L}$ and $k = \sin(\theta_0/2)$. We recognise the first term as the known formula for the period of a pendulum with small oscillations.